UM-DAE Centre for Excellence in Basic Sciences
Course Structure and Syllabus
5-year Integrated MSc - Mathematics Stream

P: Physics, M: Mathematics, C: Chemistry, B: Biology, G: General, H: Humanity,
ME: Math Elective, MPr : Math Project

(Revised) course structure and syllabus for
Mathematics for semesters III-IV
(2017-18)

Semester III

Subject Code	**Subject**	**Contact Hours per Week** Theory + Tutorials	**Credits**
M301	Foundations	[3+1]	4
M302	Analysis I	[3+1]	4
M303	Algebra I	[3+1]	4
M304	Elementary Number Theory	[3+1]	4
PM301	Classical Mechanics	[3+1]	4
H301	History and Philosophy of Science	[3+0]	3
		Lab Hours per Week	
GL301	Applied Electronics Laboratory	[4]	2
		Semester Credits	**25**

Semester IV

Subject Code	**Subject**	**Contact Hours per Week** Theory + Tutorials	**Credits**
M401	Analysis II	[3+1]	4
M402	Algebra II	[3+1]	4
M403	Discrete Mathematics	[3+1]	4
M404	Topology I	[3+1]	4
M405	Complex Analysis	[3+1]	4
G401	Statistical Techniques and Applications	[3+1]	4
H401	World Literature	[3+0]	3
		Semester Credits	**27**

The syllabus for Mathematics courses for semesters III-IV appears on the following pages.

M301 : Foundations

1. Logic : Quantifiers and negations, illustrated by examples of mathematical and non-mathematical statements.

2. Set Theory : Unions and intersections of arbitrary families, illustrated by examples. Complements. De Morgan's laws for arbitrary collection of sets. Symmetric difference. Power set of a set. Cartesian product of two sets.

3. Relations and maps :
   (a) Relations between two sets, including the case when the two sets are the same. .
   (b) Definition of a map. Composite of two maps. Injective, surjective and bijective maps and their composites. A map is bijective if and only if it is invertible.
   (c) Image and inverse image under a map. Relation between images (resp. inverse images) and set theoretic operations. Inverse images under a composite map. Clarify a common misconception: If $f : X \to Y$ is a map and $B \subseteq Y$ then the definition of $f^{-1}(B)$ does not require the existence of f^{-1}.
   (d) Equivalence relations. Lots of examples incluing fibres of map and congruence of integers modulo n.. Equivalence classes. Giving an equivalene relation on a set X is equivalent to giving a partiition of X. Quotient set. Construction of $\mathbb{Z}$ as a quotient of $\mathbb{N} \times \mathbb{N}$.

4. Cardinality:
   (a) Finite and infinite sets.
   (b) Bijection relates to same cardinality.
   (c) Countable sets. Countably infinite and uncountable sets. Examples.
   (d) Every infinite set has a proper, countably infinite subset.
   (e) Uncountability of $\mathbb{R}$ and $\mathcal{P}(\mathbb{N})$. Algebraic numbers are countable. This yields existence of transcendental numbers.
   (f) Schroeder-Bernstein theorem.

5. Partially Ordered Sets :
   (a) Concept of partial order and total order. Examples.
   (b) Upper bound. lub, lower bound, glb.
   (c) Maximum and maximal. Minimum and minimal.
   (d) Chains, Zorn's Lemma.
   (e) Lexicographic order.

6. (a) Well-ordering Principle.
   (b) Weak and Strong Principles of Mathematical Induction.
   (c) Axiom of Choice, product of an arbitrary family of sets.
   (d) Statement (without proof) of the equivalence of Axiom of Choice, Zorn's Lemma and Well-ordering Principle.

7. Additional Topics (Optional)
   (a) Dedekind's Construction of Real Numbers.
   (b) Binary, ternary, hexadecimal etc expansions of integers (and real numbers).
   (c) Cantor Sets.

References

[1] Naive Set Theory, P. Halmos.
[2] Set Theory and Logic, R. Stoll.

Beginning sections of the following books:
[3] Topology, J. Munkres.
[4] Real Analysis, Bartle and Sherbert.

M302 : Analysis I

1. Real number system: Construction via Cauchy sequences. (Note: Dedekind cuts is an optional topic in M301.)

2. Concept of a field, ordered field, examples of ordered fields, supremum, infimum. Order completeness of $\mathbb{R}$, $\mathbb{Q}$ is not order complete. Absolute values, Archimedean property of $\mathbb{R}$. The fact that $\mathbb{C}$ is a field that cannot be made into an ordered field. Denseness of $\mathbb{Q}$ in $\mathbb{R}$. Every positive real number has a unique positive n-th root.

3. Sequences: A monotone increasing sequence which is bounded above converges to its supremum. Sandwich theorem. $lim\ \left(1+\frac{1}{n}\right)^n = e$, $lim\ \sqrt[n]{n} = 1$ and $lim\, a^{\frac{1}{n}} = 1$.

4. Subsequences and Cauchy sequences: Every sequence of real numbers has a monotone subsequence. Cauchy completeness of $\mathbb{R}$; $\mathbb{Q}$ is not Cauchy complete.

5. Infinite Series: Absolute and conditional convergence. Comparison test, ratio test, root test, Abel's alternating series test. Dirichlet's test for convergence of $\Sigma a_n b_n$. Statement of Riemann's rearrangement theorem. Power series, radius of convergence, uniform convergence via examples.

6. Continuous functions: Sequential and neighbourhood definitions; sums and products of continuous functions are continuous. Intermediate value property; continuous functions on closed and bounded intervals are bounded and attain their bounds; monotone continuous functions, inverse functions. Uniform Continuity, examples and counter-examples.

7. Differentiable functions: Definition as a function infinitesimally approximable by a linear map, equivalence with Newton's ratio definition. One-sided derivatives. The O, o and $\sim$ notations with illustrative examples. Chain rule with complete proof (using the above definition). Relation between the sign of f' and local monotonicity. Proofs of Rolle's theorem, Lagrange's and Cauchy's mean value theorems. L'Hospital's rule.
   Higher order derivatives. Convex functions. Local maxima/minima, saddle points; examples of curve sketching in the plane.
   Taylor's theorem, estimation of the remainder in Taylor's theorem.
   Power series expansions of elementary functions. Validity of term by term differentiation and integration. Binomial theorem for arbitrary real coefficients. Standard example:
   $$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

8. Riemann Integration: Upper and lower Riemann sums, basic properties. Riemann integrability, $f\colon [a,b] \longrightarrow \mathbb{R}$ continuous implies f is Riemann integrable, examples of Riemann integrable functions which are not continuous on $[a,b]$. If $f\colon [a,b] \longrightarrow \mathbb{R}$ is Riemann integrable then so is $|f|$ and $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$. Cauchy-Schwarz inequality: $\left|\int fg\right| \leq \sqrt{\int f^2}\sqrt{\int g^2}$, $\left|\int fg\right| \leq (\int f^p)^{\frac{1}{p}}(\int g^q)^{\frac{1}{q}}$, where $\frac{1}{p}+\frac{1}{q}=1$. Mean value theorem for integrals.

9. (Optional, if time permits):
   Improper integrals. Cauchy's condition for the existence of improper integrals, test for convergence. Gudermannian and other examples.

References

[1] Introduction to Real Analysis: Robert G. Bartle and Donald R. Sherbert, 4th ed., Wiley Publications, 2011

[2] A First Course in Analysis: George Pedrick, Undergraduate Texts in Mathematics, SpringerScience and Business Media, 2012. ISBN: 1441985549, 9781441985545

[3] Principles of Mathematical Analysis, Walter Rudin, (Indian Edition), 3rd ed,. McGraw-Hill, 1976. ISBN: 9780070542358.

[4] Tom M. Apostol, Mathematical Analysis, 2nd ed., Pearson Education, 1974. ISBN: 9780201002881.

[5] Michael Spivak, Calculus, 4th ed., Publish or Perish, 2008. ISBN: 9780914098911.

M303 : Algebra I (Groups, rings, fields)

1. Division algorithm in $\mathbb{Z}$, fundamental theorem of arithmetic.
2. Recollection of equivalence relations and equivalence classes, illustrate by congruence classes of integers modulo n.
3. Definition of a group, examples including matrices, permutation groups, groups of symmetry, roots of unity.
4. First properties of a group, laws of exponents, finite and infinite groups.
5. Subgroups and cosets, order of an element, Lagrange theorem, normal subgroups, quotient groups.
6. Detailed look at the group S_n of permutations, cycles and transpositions, even and odd permutations, the alternating group.
7. Homomorphisms, kernel, image, isomorphism, the fundamental theorem of group homomorphisms.
8. Cyclic groups, subgroups and quotients of cyclic groups, finite and infinite cyclic groups.
9. Cayleys theorem on representing a group as a permutation group.
10. Conjugacy classes, centre, class equation, centre of a p-group.
11. (Optional, if time permits) Sylow theorems.
12. Definition of a ring, examples including congruence classes modulo n.
13. Ideals, quotient rings, homomorphisms, units, fields, non-zero divisors, integral domains, field of fractions of an integral domain.
14. Division algorithm in $K[X]$, where K is a field; $K[X]$ and $\mathbb{Z}$ are PID's.
15. (Optional, if time permits) Unique factorization domains, Gauss Lemma.

References

[1] M. Artin, Algebra, Prentice Hall of India, 1994.

[2] D.S. Dummit and R.M. Foote, Abstract Algebra, 2nd Ed., John Wiley, 2002.

[3] Joseph Gallion, Contemporary Abstract Algebra, Narosa.

[4] N. Jacobson, Basic Algebra (volumes I and II), Hindustan Publishing Corporation, 1983.

M304 : Elementary Number theory

1. Fundamental theorem of arithmetic, divisibility in integers.
2. Prime numbers and infinitude of primes. Infinitude of primes of special types. Special primes like Fermat primes, Mersenne primes, Lucas primes etc.
3. Euclidean algorithm, greatest common divisor, least common multiple.
4. Equivalence relations and the notion of congruences. Wilson's theorem and Fermat's little theorem. Chinese remainder theorem.
5. Gaussian integers.
6. Continued fractions and their applications.
7. Primitive roots, Euler's Phi function.
8. Sum of divisors and number of divisors, Möbius inversion.
9. Quadratic residues and non-residues with examples.
10. Euler's Criterion, Gauss' Lemma.
11. Quadratic reciprocity and applications.
12. Applications of quadratic reciprocity to calculation of symbols.
13. Legendre symbol: Definition and basic properties.
14. Fermat's two square theorem, Lagrange's four square theorem.
15. Pythagorean triples.
16. Diophantine equations and Bachet's equation. The duplication formula.

References

[1] D. Burton, Elementary Number Theory.

[2] Kenneth H. Rosen, Elementary number theory and its applications.

[3] Niven, Ivan M.; Zuckerman, Herbert S.; Montgomery, Hugh L, An Introduction to the Theory of Numbers.

PM301 : Classical Mechanics

Recap: Newton's laws, vector algebra, gradient; momentum, energy, constraints, conservative forces, potential energy, angular momentum. Inertial and non - inertial frames, fictitious forces, Foucault pendulum, effects of Coriolis force. Central forces, conservation of energy and angular momentum, trajectories, orbits, 1/r potential (quadrature), classical scattering, two body problem, centre of mass and relative motions. Rigid body motion, moment of inertia tensor, energy and angular momentum, Euler's theorem, motion of tops, gyroscope, motion of the Earth. Introduction to Lagrangian through variational principle, applications of variational principle.

Relativity: Historical background, inconsistency of electrodynamics with Galilean relativity. Einstein's hypothesis and Lorentz transformation formula, length contraction, time dilation, Doppler shift. Energy, momentum and mass, mass - energy equivalence. Four vector notation, consistency of electrodynamics with relativity.

References

[1] An Introduction to Mechanics, 1st Edition, D. Kleppner and R. J. Kolenkow, Tata McGraw - Hill Education, 2007

[2] Classical Mechanics, 5th Edition, T. W. B. Kibble, F. Berkshire, World Scientific 2004.

[3] Introduction to Special Relativity, R. Resnik, Wiley (India), 2012

[4] Spacetime Physics, 2nd Edition, E. F. Taylor, J. A. Wheeler, W. H. Freeman and Co. 1992.

[5] Classical mechanics, N. C. Rana, P. S. Joag, Tata McGraw-Hill Education, 2001.

M401 : Analysis II

1. Linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$, partial derivatives. Tangent plane and normal line to a surface at a point. Directional derivative. Jacobian, polar and spherical polar coordinates. Chain rule. Mean value property and Taylor's theorem for several variables.

2. Parametrized surfaces, coordinate transformations, Inverse function theorem, Implicit function theorem, Rank theorem.

3. Critical points, maxima and minima, saddle points, examples of quadric surfaces in 3-space. Lagrange multiplier method.

4. Multiple integrals, Riemann and Darboux integrals, Iterated integrals. Area and volume. Improper integrals.

5. Integration on curves and surfaces: Green's theorem, Differential forms, Gauss' Divergence theorem, Stokes' theorem.

6. (Optional, if time permits): Beta and gamma functions; $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

References

[1] Michael Spivak, Calculus on Manifolds, A Modern Approach to Classical Theorems of Advanced Calculus, Westview Press, 1965. ISBN: 0805390219.

[2] James Munkres, Analysis on Manifolda, Westiew Press, 2nd ed.,1997. ISBN: 0201315963.

[3] Wendell H. Fleming, Functions of Several Variables, Undergraduate Texts in Mathematics, 2nd Ed., Springer-Verlag, 1977.

[4] Jerrold E. Marsden, Anthony J. Tromba and Alan Weinstein, Basic Multivariable Calculus, W. H. Freeman and Co. Ltd., 2001. ISBN: 9780716724438

[5] Principles of Mathematical Analysis, Walter Rudin, (Indian Edition), 3rd ed,. McGraw-Hill, 1976. ISBN: 9780070542358.

M402 : Algebra II (Linear Algebra)

Note 1: This is essentially a first course on vector spaces. However, as modules over a general ring are needed later in several courses and as it is desirable to give students time to become comfortable with this concept, modules are already introduced in the first item of this syllabus. Emphasize that vector spaces are special cases of modules, in which case several properties are available as discussed in the remaining items.

1. Modules over a commutative ring, submodules and quotient modules, homomorphisms, fundamental theorem of module homomorphisms, exact sequences, finitely generated modules, free modules.
2. Vector spaces as modules over a field, subspaces, quotient spaces.
3. Span and linear independence, basis, dimension.
4. Linear maps and their corrrespondence with matrices with respect to given bases, change of bases.
5. Eigenvalues, eigenvectors, eigenspaces, characteristic polynomial, Cayley-Hamilton.
6. Bilinear forms, inner product spaces, Gram-Schmidt process, diagonalization, spectral theorem.
7. (Optional) Classical groups.

Note 2: Jordan and rational canonical forms to be done in M602 in Semester VI as an application of the structure of finitely generated modules over a PID.

References

[1] M. Artin, Algebra, Prentice Hall of India, 1994.

[2] D.S. Dummit and R. M. Foote, Abstract Algebra, 2nd Ed., John Wiley, 2002.

[3] K. Hoffman and R. Kunze, Linear Algebra, Prentice Hall, 1992.

[4] N. Jacobson, Basic Algebra II, Hindustan Publishing Corporation, 1983.

[5] S. Lang, Algebra, 3rd ed. Springer (India) 2004.

M403 : Discrete Mathematics

1. BASICS: Pigeonhole Principle, Elementary Counting Techniques, Permutations and Combinations, Binomial and Multinomial Thoerems, Partitions, Stirling Numbers.
2. FORMAL SERIES: Formal series, Generating functions, Formal convergence, Infinite sum and products.
3. GENERATING FUNCTIONS: Recurrences, Catalan numbers, Convolutions, Evaluating sums, Exponential formula, Partition functions, Infinite series.
4. SIEVE METHODS: Inclusion-Exclusion, Mobius inversion, Involution principle.
5. ENUMERATION OF PATTERNS: Symmetries and patterns, Burnside's Lemma, Symmetries on $\mathbb{R}$ and $\mathbb{N}$.
6. PARTITIONS AND YOUNG TABLEAUX: An Introduction to the Combinatorics of Young Tableaux.
7. ADDITIONAL TOPICS: Lattice Paths and Gaussian Coefficients. Infinite Matrices and Inversion of Sequences. Probability Generating Functions. Symmetric Polynomials and Functions. Schur Functions. RSK Algorithm. Hypergeometric Sums and Hypergeometric Series.

References

[1] Martin Aigner - *A Course in Enumeration.*

[2] W. Fulton - *Young Tableaux.*

[3] Ronald Graham, Donald Knuth, Oren Patashnik - *Concrete Mathematics.*

[4] Richard Stanley - *Enumerative Combinatorics.*

[5] Ioan Tomescu, Robert Melter - *Problems in Combinatorics and Graph Theory.*

M404 : Topology I

1. Metric spaces: Definition and basic examples including the following:

   (i) The discrete metric on any set.

   (ii) $\mathbb{R}$ and $\mathbb{R}^n$ with Euclidean metrics, Cauchy-Schwarz inequality, definition of a norm on a finite dimensional $\mathbb{R}$-vector space and the metric defined by a norm.

   (iii) The set $\mathcal{C}[0,1]$ with the metric given by $\sup|f(t) - g(t)|$ (resp. $\int_0^1 |f(t) - g(t)|dt$).

   (iv) Metric subspaces, examples.

2. Topology generated by a metric: Open and closed balls, open and closed sets, complement of an open (closed) set, arbitrary unions (intersections) of open (closed) sets, finite intersections (unions) of open (closed) sets, open (closed) ball is an open (closed) set, a set is open if and only if it is a union of open balls, Hausdorff property of a metric space.

3. Equivalence of metrics, examples, the metrics on $\mathbb{R}^2$ given by $|x_1 - y_1| + |x_2 - y_2|$ (resp. $\max\{|x_1 - y_1|, |x_2 - y_2|\}$) is equivalent to the Euclidean metric, the shapes of open balls under these metrics.

4. Limit points, isolated points, interior points, closure, interior and boundary of a set, dense and nowhere dense sets.

5. Continuous maps: ε-δ definition and characterization in terms of inverse images of open (resp. closed) sets, composite of continuous maps, pointwise sums and products of continuous maps into $\mathbb{R}$, homeomorphism, isometry, an isometry is a homeomorphism but not conversely, uniformly continuous maps, examples.

6. Complete metric spaces: Cauchy sequences and convergent sequences, a subspace of a complete metric space is complete if and only if it is closed, Cantor intersection theorem, Baire category theorem and its applications, completion of a metric space.

7. General topological spaces, stronger and weaker topologies, continuous maps, homeomorphisms, bases and subbases, finite products of topological spaces.

8. Basic separation axioms and first and second countability axioms.

9. Compactness for general topological spaces: Finite subcoverings of open coverings and finite intersection property, continuous image of a compact set is compact, compactness and Hausdorff property.

10. Compactness for metric spaces: Bolzano-Weierstrass property, the Lebesgue number for an open covering, sequentially compact and totally bounded metric spaces, Heine-Borel theorem, compact subsets of $\mathbb{R}$, a continuous map from a compact metric space is uniformly continuous.

11. Connectedness: definition, continuous image of a connected set is connected, characterization in terms of continuous maps into the discrete space $\mathbb{N}$, connected subsets of $\mathbb{R}$, intermediate value theorem as a corollary, countable (arbitrary) union of connected sets, connected components,

References

[1] E. T. Copson, *Metric spaces.*

[2] M. Eisenberg, *Topology.*

[3] R.H. Kasriel, *Undergraduate topology.*

[4] W. Rudin, *Principles of mathematical analysis.*

[5] G. F. Simmons, *Topology and modern analysis.*

[6] W. A. Sutherland, *Introduction to metric and topological spaces.*

M405 : Complex Analysis

1. Complex numbers and Riemann sphere. Möbius transformations.
2. Analytic functions. Cauchy-Riemann conditions, harmonic functions, Elementary functions, Power series, Conformal mappings.
3. Contour integrals, Cauchy theorem for simply and multiply connected domains. Cauchy integral formula, Winding number.
4. Moreras theorem. Liouvilles theorem, Fundamental theorem of Algebra.
5. Zeros of an analytic function and Taylors theorem. Isolated singularities and residues, Laurent series, Evaluation of real integrals.
6. Zeros and Poles, Argument principle, Rouchs theorem.

References

[1] L.Ahlfors, Complex Analysis.

[2] R.V. Churchill and J. W. Brown, Complex Variables and Applications, International Student Edition,Mc-Graw Hill, 4th ed., 1984.

[3] B.R. Palka, An Introduction to Complex Function Theory, UTM Springer-Verlag, 1991.

[4] Donald Sarason, Notes on Complex Function Theory, HBA.